Fluid Behavior in a Longitudinally Excited, Cylindrical Tank of Arbitrary Sector-Annular Cross Section

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An analytic study is made of the fluid motion in a rigid container subjected to a harmonic, longitudinal excitation. At low frequencies of excitation the free surface of the fluid will either maintain its initial, planar shape or develop a steady-state oscillation. Previous analyses suggest that this steady-state response can be a subharmonic, a harmonic, or a superharmonic of the excitation frequency. Accompanying experiments, however, failed to produce the proper harmonic or superharmonic oscillations. The present paper gives an explanation for this failure. Furthermore, a general description of the fluid motion in a cylindrical tank with an arbitrary sector-annular cross section is obtained.

1. Introduction

FOR the design of large, liquid propellant vehicles, a thorough understanding of fluid motion in moving containers is necessary. Although by no means a new problem, the ever-increasing size of booster rockets has served to generate much theoretical and experimental research in this area during the last decade. An excellent synopsis of the field is available in a monograph edited by Abramson.[1] Of the various types of simple tank excitation, fluid behavior due to longitudinal forcing has received the least attention. This is because a nonlinear formulation is required to produce even a quantitative description of the nonplanar motion.

Apparently, the first study of the fluid response in a vertically-forced container was made by Faraday.[2] In a series of experiments, he observed standing waves with a frequency one-half that of the container. Similar experiments were later conducted by Matthiessen.[3,4] However, he reported the liquid vibrations were of the same frequency as the container. To solve the disagreement, Lord Rayleigh[5] repeated the experiments of Faraday. He, likewise, observed a one-half subharmonic oscillation of the free surface.

An analytic study of the problem was published in 1954 by Benjamin and Ursell.[6] Using a linear formulation, they obtained the stability boundaries for the planar free surface. This analysis was not adequate to predict the finite-amplitude motion occurring after the initial, planar shape becomes unstable; however, it did suggest that the nonplanar motion could have any frequency $N/2$ times that of the excitation, where N is a positive integer. Since this includes both the one-half subharmonic response and the harmonic response, the apparent disagreement of the experimental results of Matthiessen and Faraday was explained. Experiments conducted by Benjamin and Ursell, however, produced only the one-half subharmonic response.

Skalak and Yarymovych[7] investigated the fluid motion in an infinitely deep, rectangular tank subjected to harmonic excitation along the longitudinal axis. They succeeded in obtaining an analytic description of the nonplanar surface motion using a nonlinear formulation and solution procedures developed by Penney and Price.[8] The analysis indicated that this motion could be a one-half subharmonic, a harmonic, or a superharmonic of the excitation. Again, only the subharmonic was produced experimentally.

The fluid motion in a finite-deep tank of circular cross section was investigated, both analytically and experimentally, by Dodge, Kana, and Abramson.[9] Both one-half subharmonic and harmonic surface responses were observed in their experiments. However, because of the character of the harmonic response, they suggested it had some origin other than the instability of the planar free surface. Damping was thought to prevent the occurence of proper harmonic and superharmonic oscillations.

Chu[10] has recently given a general formulation for the subharmonic oscillations in an arbitrary tank, following the approach indicated by Moiseev.[11] Particular results were obtained for a rectangular tank and found to be in good agreement with those of Skalak and Yarymovych. No consideration was given to the stability of the nonplanar, steady-state motions predicted by the analysis.

This paper presents an analytic study of the fluid behavior in a longitudinally-excited, cylindrical tank with an arbitrary sector-annular cross section and finite depth. A description of the nonplanar response based on the assumption of no coupling between modes is determined and compared with the results of a coupled analysis. An investigation of the stability of the steady-state motions is also discussed. It is shown that in the usual experiment the degree of stability of the harmonic and superharmonic responses is normally inadequate to permit their development, even with an inviscid fluid.

2. Basic Equations

The problem will be formulated with respect to the tank-fixed, polar coordinate system shown in Fig. 1. The spatial variable z is measured from the undisturbed free surface, along the axis of the tank; $Z(t)$ represents the specified tank excitation; and $\bar{Z}(r,\theta,t)$ denotes the shape of the free surface.

The fluid is assumed to be incompressible, inviscid, and irrotational. Hence, the fluid velocity can be represented

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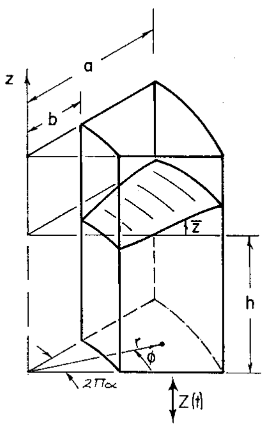

Fig. 1 Tank nomenclature and coordinate system.

as the gradient of a potential function $\Phi(r,\theta,z,t)$,

$$\bar{v} = \nabla\Phi \qquad (1)$$

Furthermore, the velocity potential must satisfy Laplace's equation within the interior of the fluid,

$$\nabla^2\Phi = 0; \quad b < r < a,\ 0 < \phi < 2\pi\alpha,\ h < z < \bar{Z} \qquad (2)$$

With the assumption that the tank is rigid and impermeable, the wall conditions may be written as

$$\Phi_r = 0; \quad r = b,a \qquad (3)$$

$$\Phi_\phi = 0; \quad \phi = 0,2\pi\alpha \qquad (4)$$

$$\Phi_z = 0; \quad z = -h \qquad (5)$$

Subscripts denote partial differentiation.

Two free surface conditions must be satisfied. The first, referred to as the kinematic condition, results from the requirement that a particle on the surface and the surface itself have same velocity;

$$\bar{Z}_t + \nabla\Phi\cdot\nabla\bar{Z} = \Phi_z; \quad z = \bar{Z} \qquad (6)$$

A dynamic condition is obtained from Bernoulli's unsteady flow equation. If g represents the gravitational acceleration, this condition is

$$\Phi_t + (g + \ddot{Z})z + \tfrac{1}{2}(\nabla\Phi\cdot\nabla\Phi) = 0; \quad z = \bar{Z} \qquad (7)$$

The dots represent time differentiation.

Equations (2–7) formulate the boundary value problem. A potential function which satisfies the Laplace equation and the wall conditions is found by separation of variables to be

$$\Phi = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} A_{mn}(t)\cdot\cos\frac{m\phi}{2\alpha}\cdot\frac{\cosh[\zeta_{mn}(z/a + h/a)]}{\cosh(\zeta_{mn}\cdot h/a)}\times C_{m/2\alpha}\left(\zeta_{mn}\cdot\frac{r}{a}\right) \qquad (8)$$

The cylinder function is defined in terms of Bessel functions of order $m/2\alpha$ as follows:

$$C_{m/2\alpha}(\zeta_{mn}\cdot r/a) = J_{m/2\alpha}(\zeta_{mn}\cdot r/a) - J'_{m/2\alpha}(\zeta_{mn})\cdot Y_{m/2\alpha}(\zeta_{mn}\cdot r/a)/Y'_{m/2\alpha}(\zeta_{mn}) \qquad (9)$$

Prime denotes differentiation with respect to the argument and ζ_{mn} is the nth positive root of

$$J'_{m/2\alpha}(\zeta\cdot b/a) - J'_{m/2\alpha}(\zeta)\cdot Y'_{m/2\alpha}(\zeta\cdot b/a)/Y'_{m/2\alpha}(\zeta) = 0 \qquad (10)$$

Roots corresponding to various tank geometries have been tabulated in Ref. 12.

The free surface elevation is now expanded in a Fourier-Bessel series,

$$\bar{Z}(r,\phi,t) = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} a_{mn}(t)\cdot\cos\frac{m\phi}{2\alpha}\cdot C_{m/2\alpha}\left(\zeta_{mn}\cdot\frac{r}{a}\right) \qquad (11)$$

The time functions A_{mn} and a_{mn} must be chosen so as to satisfy the free surface conditions, Eqs. (6) and (7). This cannot be directly achieved, however, since the conditions are nonlinear and apply at an unknown surface. An equivalent system will now be developed.

In connection with the dynamic free surface condition, consider the function

$$E(r,\phi,z,t) = \Phi_t + (g + \ddot{Z})z + \tfrac{1}{2}(\nabla\Phi\cdot\nabla\Phi) \qquad (12)$$

which assumes a value of zero at $\bar{Z}$. Expansion of E in a Taylor series about $z = 0$, evaluating at $\bar{Z}$, and substitution of Eqs. (8) and (11) into the result yields

$$\sum_{m=0}^{\infty}\sum_{n=1}^{\infty} [\dot{A}_{mn} + (g + \ddot{Z})a_{mn}]\cos\frac{m\phi}{2\alpha}\cdot C_{m/2\alpha}\left(\zeta_{mn}\cdot\frac{r}{a}\right) + F(r,\phi,A_{01},A_{11},\ldots,\dot{A}_{01},\dot{A}_{11},\ldots,a_{01},a_{11},\ldots) = 0 \qquad (13)$$

The function F contains products of A_{mn}, $\dot{A}_{mn}$, and a_{mn}; hence, it is nonlinear in the dependent variables. Let F be formally expanded in a Fourier-Bessel series,

$$F = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} f_{mn}(A_{01},\ldots,\dot{A}_{01},\ldots,a_{01},\ldots)\cos\frac{m\phi}{2\alpha}\cdot C_{m/2\alpha}\left(\zeta_{mn}\cdot\frac{r}{a}\right)$$

Substitution into Eq. (13) reduces the dynamic free surface condition to

$$\dot{A}_{mn} + (g + \ddot{Z})a_{mn} + f_{mn} = 0 \qquad (14)$$

$$m = 0,1\ldots; \quad n = 1,2,\ldots$$

A similar manipulation of the kinematic condition provides

$$\dot{a}_{mn} - (\zeta_{mn}/a)\cdot\tanh(\zeta_{mn}\cdot h/a)A_{mn} + h_{mn} = 0 \qquad (15)$$

$$m = 0,1,\ldots; \quad n = 1,2,\ldots$$

The function h_{mn} is also nonlinear in the dependent variables, containing products of A_{mn}, a_{mn}, and $\dot{a}_{mn}$.

Solution of the problem would be completed by satisfying Eqs. (14) and (15), along with the appropriate initial conditions. With the exception of the identically zero response discussed in the next section, an exact solution is not likely to be achieved; these equations describe an infinite system of coupled, nonlinear differential equations. This formulation of the free surface conditions has considerable usefulness, however, since order assumptions, leading to approximate solutions, may be readily introduced.

3. Identically Zero Response

Unlike other types of tank excitation, this problem has an identically zero, steady-state solution. This corresponds to the free surface maintaining a planar shape. The occurrence of this response in reality, however, depends on its stability.

Let δa_{mn} represent a perturbation about the steady-state motion. Equations (14) and (15) combine to yield

$$\delta\ddot{a}_{mn} + (\omega_{mn}^2/g)(g + \ddot{Z})\delta a_{mn} = 0$$

Here ω_{mn} is the natural frequency of the (m,n) mode and is given by

$$\omega_{mn}^2 = (g/a)\zeta_{mn}\tanh(\zeta_{mn}h/a)$$

For harmonic excitation, $Z = Z_0\cos\Omega t$, Mathieu's equation is obtained,

$$d^2(\delta a_{mn})/d\tau^2 + (\lambda_{mn} - 2\beta_{mn}\cos\tau)\delta_{mn} = 0 \qquad (16)$$

where $\tau = \Omega t$ and the parameters are defined by

$$\lambda_{mn} = \omega_{mn}^2/\Omega^2$$

$$\beta_{mn} = Z_0/a/2g/a \cdot \omega_{mn}^2$$

For the planar free surface to be stable, the solutions of each of Eqs. (16) must remain bounded. Since the character of the solutions is completely determined by the parameters λ_{mn} and β_{mn}, consideration need only be given to their location in the parametric plane shown in Fig. 2. This plane is separated into stable and unstable regions. The cross-hatched areas denote unstable regions and the boundaries are given by

$$\begin{aligned}
a_1&: \ \lambda = \tfrac{1}{4} + \beta - \beta^2/2 - \beta^3/4 + 0(\beta^4)\\
b_1&: \ \lambda = \tfrac{1}{4} - \beta - \beta^2/2 + \beta^3/4 + 0(\beta^4)\\
a_2&: \ \lambda = 1 + 5\beta^2/3 + 0(\beta^4)\\
b_2&: \ \lambda = 1 - \beta^2/3 + 0(\beta^4)\\
a_3&: \ \lambda = \tfrac{9}{4} + \beta^2/4 - \beta^3/4 + 0(\beta^4)\\
b_3&: \ \lambda = \tfrac{9}{4} + \beta^2/4 - \beta^3/4 + 0(\beta^4)\\
&\cdots\cdots\cdots\cdots\\
a_r&: \ \lambda = (r/2)^2 + 2\beta^2/(r^2 - 1) + 0(\beta^4)\\
b_r&: \ \lambda = (r/2)^2 + 2\beta^2/(r^2 - 1) + 0(\beta^4)
\end{aligned} \tag{17}$$

Corresponding to each mode (m,n) there is a point $(\lambda_{mn},\beta_{mn})$ located on the line

$$\beta = [(Z_0/a)/(2g/a)]\cdot\Omega^2\lambda \tag{18}$$

For stability it is necessary that each member of this infinite sequence of points fall into a stable region of the parametric plane. If one or more fall into unstable regions, the disturbances grow until the free surface breaks up, or until a finite-amplitude, steady-state motion is attained.

The effect of damping is to prevent the unstable regions from extending to the λ axis. It is suggested that in most real fluids there is sufficient damping such that except for the first several unstable regions, all others will be located completely above the line described by Eq. (18). Hence, consideration need only be given to those modes with frequencies in the lower ranges.

From the theory of Mathieu's equation (see, for example, Ref. 13), it is known that the oscillatory part of a solution with parameters which lie in unstable region N is dominated by a term having frequency $N\Omega/2$. This is the basis for Benjamin and Ursell's conclusion that a nonplanar response could be excited with any frequency $N\Omega/2$.

4. Finite-Amplitude Response

In this section, approximate solutions to Eqs. (14) and (15), representing a finite-amplitude fluid response, will be found. Experiments have shown that when the planar free surface becomes unstable, bounded and periodic oscillations can develop. Since a linear theory predicts unbounded, exponentially increasing motion, nonlinear terms must be retained in Eqs. (14) and (15) to describe this phenomenon analytically.

The investigation of this finite-amplitude motion is made tractable by the introduction of simplifying assumptions. Order assumptions are made on A_{mn} and a_{mn}, and the formulation is restricted to terms of third order or less. This allows closed form approximations for f_{mn} and h_{mn} to be found, and reduces the system described by Eqs. (14) and (15) to a finite number of equations. Various steady-state solutions are then determined through use of the Rayleigh-Ritz procedure.

As a first approximation, assume there is no coupling of modes. This is found to give an excellent quantitative description of the fluid behavior. If the fluid responds in the (i,j) mode, the velocity potential and the free surface shape

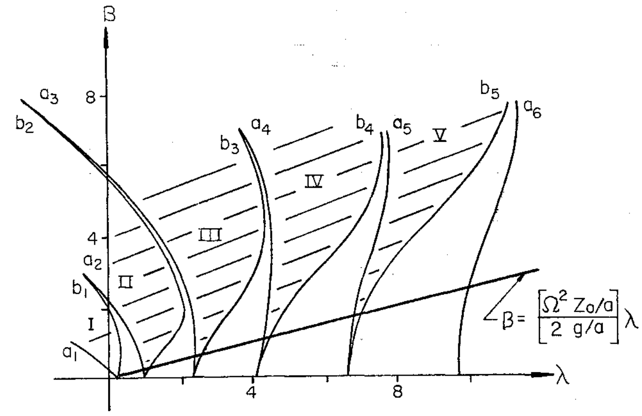

Fig. 2 Parametric plane for Mathieu's equation.

are given by

$$\Phi = A_{ij}\cdot\cos\frac{i\phi}{2\alpha}\cdot\frac{\cosh[\zeta_{ij}(z/a + h/a)]}{\cosh(\zeta_{ij}h/a)}\cdot C_{i/2\alpha}\left(\zeta_{ij}\frac{r}{a}\right)$$

$$\bar{Z} = a_{ij}\cdot\cos\frac{i\phi}{2\alpha}\cdot C_{i/2\alpha}\left(\zeta_{ij}\frac{r}{a}\right)$$

Uncoupled Motion in the (1,1) Mode

Because of the importance in vehicle design, motion in the first antisymmetric mode is studied in detail. The assumption of no coupling corresponds to assuming A_{11} and a_{11} are of first order, whereas all other unknown time functions are of third or higher order.

The function F occurring in Eq. (13) becomes

$$\begin{aligned}
F = {} & A_{11}^2\left\{\frac{1}{2}\left(\frac{\zeta_{11}}{a}\right)^2\cdot C_{1/2\alpha}'^2\left(\zeta_{11}\frac{r}{a}\right)\cdot\cos^2\left(\frac{\phi}{2\alpha}\right) + \right.\\
& \left(\frac{1}{2\alpha}\right)^2\cdot\frac{1}{r^2}\cdot C_{1/2\alpha}^2\left(\zeta_{11}\frac{r}{a}\right)\cdot\sin^2\left(\frac{\phi}{2\alpha}\right) + \left(\frac{\zeta_{11}}{a}\right)^2\times\\
& \left.\tanh^2\left(\zeta_{11}\frac{h}{a}\right)\cdot C_{1/2\alpha}^2\left(\zeta_{11}\frac{r}{a}\right)\cdot\cos^2\left(\frac{\phi}{2\alpha}\right)\right\} + \\
& \dot{A}_{11}a_{11}\left\{\left(\frac{\zeta_{11}}{a}\right)\cdot\tanh\left(\zeta_{11}\frac{h}{a}\right)\cdot C_{1/2\alpha}^2\left(\zeta_{11}\frac{r}{a}\right)\cdot\cos^2\left(\frac{\phi}{2\alpha}\right)\right\} + \\
& A_{11}^2a_{11}\left\{\left(\frac{\zeta_{11}}{a}\right)^3\cdot\tanh\left(\zeta_{11}\frac{h}{a}\right)\cdot C_{1/2\alpha}'^2\left(\zeta_{11}\frac{r}{a}\right)\times\right.\\
& C_{1/2\alpha}\left(\zeta_{11}\frac{r}{a}\right)\cdot\cos^3\left(\frac{\phi}{2\alpha}\right) + \left(\frac{1}{2\alpha}\right)^2\cdot\frac{1}{r^2}\cdot\left(\frac{\zeta_{11}}{a}\right)\cdot\tanh\left(\zeta_{11}\frac{h}{a}\right)\times\\
& C_{1/2\alpha}^3\left(\zeta_{11}\frac{r}{a}\right)\cdot\sin^2\left(\frac{\phi}{2\alpha}\right)\cdot\cos\left(\frac{\phi}{2\alpha}\right) + \left(\frac{\zeta_{11}}{a}\right)^3\cdot\tanh\left(\zeta_{11}\frac{h}{a}\right)\times\\
& \left.C_{1/2\alpha}^3\left(\zeta_{11}\frac{r}{a}\right)\cdot\cos^3\left(\frac{\phi}{2\alpha}\right)\right\} + \dot{A}_{11}a_{11}^2\left\{\frac{1}{2}C_{1/2\alpha}^3\times\right.\\
& \left.\left(\zeta_{11}\frac{r}{a}\right)\cdot\cos^3\left(\frac{\phi}{2\alpha}\right)\right\}
\end{aligned} \tag{19}$$

From this the function appearing in Eq. (14) is found to be

$$f_{11} = d_1\tanh(\zeta_{11}h/a)A_{11}^2a_{11}/a^3 + d_2\dot{A}_{11}a_{11}^2/a^2 \tag{20}$$

A similar procedure shows the nonlinear function in the kinematic free surface condition Eq. (15) to be

$$h_{11} = g_1\tanh(\zeta_{11}h/a)A_{11}a_{11}^2/a^3 \tag{21}$$

The parameters d_1, d_2, and g_1 depend on the sector angle parameter α and the ratio of inner to outer radius. They were obtained by numerical integration and are listed in Table 1.

Table 1 Coefficients in the reduced free surface conditions for the (1,1) mode

	$\alpha = \frac{1}{2}$			$\alpha = \frac{1}{4}$			$\alpha = \frac{1}{8}$		
b/a	d_1	d_2	g_1	d_1	d_2	g_1	d_1	d_2	g_1
0	1.7403	0.3544	0.8841	5.2712	0.6472	2.8165	17.9921	1.2688	10.0533
0.1	1.6702	0.3473	0.8689	5.2660	0.6468	2.8150	17.9917	1.2688	10.0531
0.2	1.5220	0.3347	0.8442	5.1982	0.6423	2.7984	17.9897	1.2687	10.0524
0.3	1.3773	0.3265	0.8120	5.0194	0.6341	2.7783	17.9442	1.2665	10.0368
0.4	1.2401	0.3181	0.7568	4.8692	0.6424	2.8301	17.7055	1.2570	9.9764
0.5	1.1017	0.3050	0.6828	4.8150	0.6734	2.9171	17.4628	1.2654	10.1078
0.6	0.9658	0.2870	0.6020	4.7040	0.7011	2.9101	18.2907	1.3802	11.0332
0.7	0.8334	0.2659	0.5237	4.4411	0.7048	2.7687	19.8163	1.5774	12.2638
0.8	0.7237	0.2438	0.4523	4.0479	0.6818	2.5289	20.3980	1.7187	12.7288

By the elimination of A_{11}, the two free surface conditions may be combined to yield

$$\ddot{\alpha}_{11} + \omega_{11}^2\left(1 - \frac{Z_0\Omega^2}{g}\cos\Omega t\right)(\alpha_{11} - m_1\alpha_{11}^3) + m_2\ddot{\alpha}_{11}\alpha_{11}^2 + m_3\alpha_{11}\dot{\alpha}_{11}^2 = 0 \quad (22)$$

where

$$\alpha_{mn} = a_{mn}/a$$

and

$$m_1 = 2g_1/\zeta_{11}$$
$$m_2 = d_2 - g_1/\zeta_{11} \quad (23)$$
$$m_3 = (d_1 + 2g_1)/\zeta_{11}$$

Approximate steady-state solutions are found by using the Rayleigh-Ritz procedure. For the one-half subharmonic response assume

$$\alpha_{11} = \zeta\sin\omega t + \eta\cos\omega t \quad (24)$$

in which $\omega = \Omega/2$. Two steady-state responses are obtained,

$$\eta = 0$$

$$r^2 = \left(\frac{\omega}{\omega_{11}}\right)^2 = \frac{1 - \frac{3}{4}m_1\zeta^2}{1 - 2\zeta_{11}\tanh(\zeta_{11}h/a)\cdot(Z_0/a)\cdot(1 - m_1\zeta^2) + \frac{1}{4}(3m_2 - m_3)\zeta^2} \quad (25)$$

and

$$\zeta = 0$$

$$r^2 = \frac{1 - \frac{3}{4}m_1\eta^2}{1 + 2\zeta_{11}\tanh(\zeta_{11}h/a)\cdot(Z_0/a)\cdot(1 - m_1\eta^2) + \frac{1}{4}(3m_2 - m_3)\eta^2} \quad (26)$$

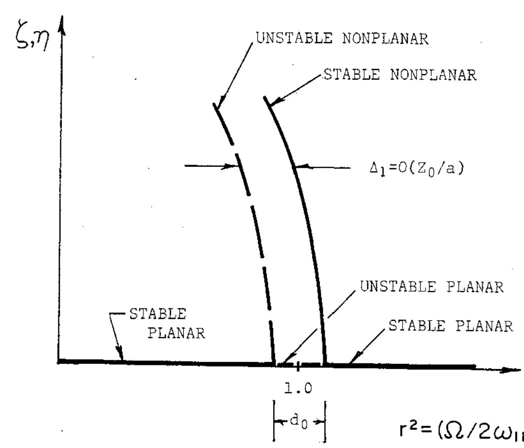

Fig. 3 One-half subharmonic response.

In these and the following equations, r represents the ratio of the response frequency and the natural frequency.

The stability of these periodic motions was investigated using the method of Andronov and Witt.[14] The first solution was found to be stable and the second is unstable. A schematic representation of the two responses is presented in Fig. 3.

Although given little consideration in the past, the curve representing the unstable solution is of extreme importance. It serves as a boundary separating the region in which transient motion will seek the finite-amplitude steady-state motion and the region in which transient motion will reduce to the planar free surface. If, for example, a steady-state motion corresponding to a point on the solid curve is attained and the fluid is then given a disturbance, it is possible to "jump" past the dashed curve and the free surface return to the planar shape. The magnitude required for such a disturbance depends on the distance Δ_1 between the stable and unstable steady-state solutions.

From Eqs. (25) and (26), the distance is found to be of order (Z_0/a). This can also be determined from examination of the stability diagram for Mathieu's equation. Skalak and Yarymovych have shown that the distance d_0 appearing in Fig. 3 corresponds to the width of the first unstable region in the stability diagram. From the expressions given in Eqs. (17) for the curves a_1 and b_1, this is seen to be of order (Z_0/a). Since Δ_1 is always less than or equal to d_0, Δ_1 is also of order (Z_0/a).

Approximations for the harmonic and superharmonic solutions of Eq. (22) may also be determined using the Rayleigh-Ritz procedure. The response curves are analogous to those shown in Fig. 3. The crucial difference, however, is that for a response having a frequency $N/2$ times the excitation frequency, the distance between the stable and unstable solutions is of order $(Z_0/a)^N$. Since, in general, experimenters must resort to small excitation amplitudes to prevent the free surface from disintegrating, the separation of the solution curves is very small for the harmonic and superharmonic responses. Therefore, slight irregularities in the experimental apparatus would prevent the development of these steady-state responses.

Due to the third-order terms appearing in Eq. (22), the one-fourth subharmonic is also a solution. Let X represent the fluid amplitude and $r = \Omega/4\omega_{11}$. A Rayleigh-Ritz pro-

Table 2 Coefficients in the reduced free surface conditions for the (0,1) mode

b/a	k_1	k_2	k_3	k_4	h_1	h_2
0	3.8791	31.0462	1.3498	3.0384	−2.5861	−3.8808
0.1	3.6603	29.6604	1.2384	2.8223	−2.4402	−3.7075
0.2	3.5510	32.6007	1.1178	2.8862	−2.3673	−4.0751
0.3	4.5716	66.0616	1.2953	5.2644	−3.0477	−8.2576
0.5	−3.3243	58.2846	−0.6933	3.4188	2.2162	−7.2856
0.6	−4.4039	141.0021	−0.7405	6.6678	2.9359	−17.6251
0.7	3.0055	96.8428	0.3808	3.4514	−2.0037	−12.1053
0.8	4.0679	322.6511	0.3437	7.6827	−2.6843	−40.0716

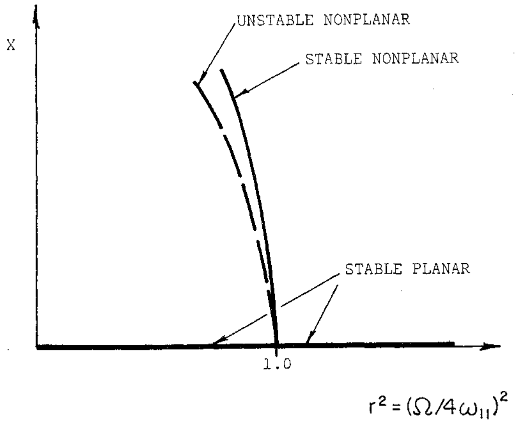

Fig. 4 One-fourth subharmonic response.

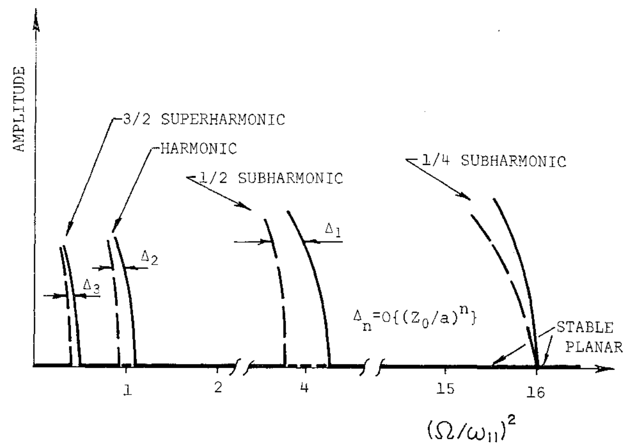

Fig. 5 Complete (1,1) mode response.

cedure yields the two steady-state solution described by

$$\alpha_{11} = X \sin\omega t \text{ or } X \cos\omega t$$

$$r^2 = \frac{1 - \frac{3}{4}m_1X^2}{1 - 2\zeta_{11}\tanh(\zeta_{11}h/a)\cdot(Z_0/a)\cdot m_1X^2 + \frac{1}{4}(3m_2 - m_3)X^2} \tag{27}$$

and

$$\alpha_{11} = X\cos(\omega t - \pi/4)$$

$$r^2 = \frac{1 - \frac{3}{4}m_1X^2}{1 + 2\zeta_{11}\tanh(\zeta_{11}h/a)\cdot(Z_0/a)\cdot m_1X^2 + \frac{1}{4}(3m_2 - m_3)X^2} \tag{28}$$

The second solution represents unstable, steady-state motion, whereas the first solution corresponds to a stable motion for

$$\frac{Z_0}{a} < \frac{3m_1 + (3m_2 - m_3)}{8m_1\zeta_{11}\tanh(\zeta_{11}h/a)} \tag{29}$$

If the excitation amplitude is sufficiently large for the inequality not to be satisfied, the fluid cannot respond in the one-fourth subharmonic.

A schematic representation of the response curves described by Eqs. (27) and (28) is given in Fig. 4. In contrast to the one-half subharmonic, these curves join at a zero amplitude. This, of course, gives an explanation for the failure of the investigation of the stability of the planar free surface to suggest the one-fourth subharmonic motion.

As discussed by Skalak and Yarymovych, the stable and unstable response curves would join at a finite amplitude if damping were included. This is referred to as the cut-off amplitude since there can be no one-fourth subharmonic response with a smaller amplitude.

Higher order terms in Eq. (22) would allow as solutions lower subharmonics than the one-fourth. As the frequencies decrease, however, the corresponding cut-off amplitudes increase, thereby lessening their physical importance.

Figure 5 shows the total response obtained from a third order analysis of a fluid which is assumed to oscillate only in the (1,1) mode. The response for any real fluid would be a doubly-infinite superposition of such plots, one for each mode.

Uncoupled Motion in the (0,1) Mode

If the fluid is assumed to respond only in the (0,1) mode and terms of higher than three are neglected, the free surface conditions become

$$\dot{A}_{01} + (g + \ddot{Z})a_{01} + k_1\tanh^2(\zeta_{01}h/a)\cdot A_{01}^2/a^2 + k_2\tanh(\zeta_{01}h/a)\cdot A_{01}^2a_{01}/a^2 + k_3\tanh(\zeta_{01}h/a)\cdot \dot{A}_{01}a_{01}/a + k_4\dot{A}_{01}a_{01}^2/a^2 = 0$$

$$\dot{a}_{01} - (\zeta_{01}/a)\cdot\tanh(\zeta_{01}h/a)A_{01} + h_1A_{01}a_{01}/a^2 + h_2\tanh(\zeta_{01}h/a)\cdot A_{01}a_{01}^2/a^3 = 0 \tag{30}$$

The constants have been determined numerically and are tabulated in Table 2. Unlike those appearing in the (1,1) mode analysis, these do not depend on the sector angle.

Elimination of A_{01} yields

$$\ddot{\alpha}_{01} + \omega_{01}^2\left(1 - \frac{Z_0\Omega^2}{g}\cos\Omega t\right)(\alpha_{01} - q_1\alpha_{01}^2 - q_2\alpha_{01}^3) + q_3\dot{\alpha}_{01}^2 + q_r\alpha_{01}\ddot{\alpha}_{01} + q_5\alpha_{01}^2\ddot{\alpha}_{01} + q_6\alpha_{01}\dot{\alpha}_{01}^2 = 0 \tag{31}$$

where

$$\begin{aligned}
q_1 &= 2h_1/[\zeta_{01}\tanh(\zeta_{01}h/a)]\\
q_2 &= 2h_2/\zeta_{01} - h_1^2/[\zeta_{01}\tanh(\zeta_{01}h/a)]^2\\
q_3 &= [h_1 + k_1\tanh^2(\zeta_{01}h/a)]/[\zeta_{01}\tanh(\zeta_{01}h/a)]\\
q_4 &= k_3\tanh(\zeta_{01}h/a) - h_1/[\zeta_{01}\tanh(\zeta_{01}h/a)]\\
q_5 &= k_4 - (h_2 + h_1k_3)/\zeta_{01}\\
q_6 &= (k_2 + h_1k_3 + 2h_2)/\zeta_{01}
\end{aligned} \tag{32}$$

The steady-state solutions of the preceding equation are again investigated using the Rayleigh-Ritz procedure. For the one-half subharmonic, the solution is assumed in the form

$$\alpha_{01} = \zeta\sin\omega t + \eta\cos\omega t \tag{33}$$

where $\omega = \Omega/2$. Two steady-state solutions are found,

$$\eta = 0$$

$$r^2 = \left(\frac{\Omega}{2\omega_{01}}\right)^2 = \frac{1 - \frac{3}{4}q_2\zeta^2}{1 - 2\zeta_{01}\tanh(\zeta_{01}h/a)\cdot(Z_0/a)(1 - q_2\zeta^2) + \frac{1}{4}(3q_5 - q_6)\zeta^2} \tag{34}$$

and

$$\zeta = 0$$

$$r^2 = \frac{1 - \frac{3}{4}q_2\eta^2}{1 + 2\zeta_{01}\tanh(\zeta_{01}h/a)\cdot(Z_0/a)\cdot(1 - q_2\eta^2) + \frac{1}{4}(3q_i - q_z)\eta^2} \tag{35}$$

These solutions have the same form as the response shown in Fig. 3. The first solution corresponds to the stable branch and the second to the unstable branch. As before, the distance between the curves is of order (Z_0/a).

Equation (31) also has harmonic and superharmonic solutions. For a response frequency equal to $N\Omega/2$, the distance

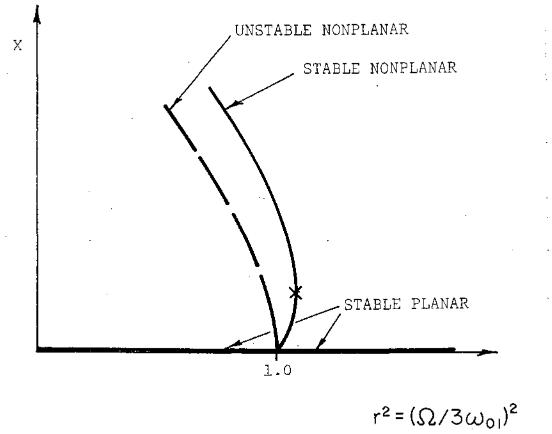

Fig. 6 One-third subharmonic response in the (0,1) mode.

between the stable and unstable branches is again of order $(Z_0/a)^N$.

Due to the nonlinear terms in Eq. (31), a one-fourth subharmonic motion is possible. Its characteristics are analogous to such a response in the (1,1) mode and will not be discussed further.

Unlike the (1,1) mode, however, a one-third subharmonic response in the (0,1) mode is possible. The two steady-state solutions are

$$\alpha_{01} = X\cdot\cos\Omega/3t = X\cdot\cos\omega t$$
$$r^2 = \frac{1 - \frac{3}{4}q_2X^2}{1 - \frac{9}{4}\zeta_{01}\tanh(\zeta_{01}\,h/a)\cdot(Z_0/a)\cdot q_1X + \frac{1}{4}(3q_5 - q_6)X^2} \tag{36}$$

and

$$\alpha_{01} = X\cdot\cos(\omega t - \pi/3)$$
$$r^2 = \frac{1 - \frac{3}{4}q_2X^2}{1 + \frac{9}{4}\zeta_{01}\tanh(\zeta_{01}\,h/a)\cdot(Z_0/a)\cdot q_1X + \frac{1}{4}(3q_5 - q_6)X^2} \tag{37}$$

A schematic representation of these solutions is presented in Fig. 6. The left branch represents Eq. (37) and is always unstable. The right branch represents Eq. (36) and is stable only above the vertical tangent. This is of interest since it shows the existence of a "cut-off" amplitude for the one-third subharmonic response even for an inviscid fluid. As noted previously, the retention of higher order terms in the governing equations would allow additional subharmonics as solutions.

Coupled Motion

In reality, it is not possible to completely isolate the linear modes. However, it is possible to develop motion which is dominated by a particular mode, with other modes having a secondary effect. This was illustrated in the experiments performed by Dodge, Kana, and Abramson[9] with a longitudinally excited cylindrical tank having a complete circular cross section. They obtained a one-half subharmonic response composed primarily of the (1,1) mode. The (0,1) and (2,1) modes were observed as secondary. For motion composed primarily of the (0,1) mode, the (0,2) appeared as secondary.

In the present work the effect of this coupling was investigated analytically by the introduction of appropriate order assumptions into the free surface Eqs. (14) and (15). In particular, for motion composed mainly of the first antisymmetric mode, the time functions a_{11} and A_{11} are assumed to be of first order, and those corresponding to the (0,1) mode and (2,1) mode are assumed to be of second order. All other time functions are taken as higher order. For motion composed primarily of the first symmetric mode, a_{01} and A_{01} are assumed first order, and a_{02} and A_{02} are taken as second order. The other modes are assumed to be of higher order.

In each case, Eqs. (14) and (15) reduce to a finite number of equations. Although the computational difficulties increase, the procedure for obtaining approximate solutions is identical to that utilized in the uncoupled analysis. For this reason, the coupled analysis is not reproduced here. Details may be found in Ref. 15.

It is of importance to note that the coupled analysis uncovered no qualitative difference in the fluid response from that determined with the uncoupled analysis. The quantitative difference is discussed in the next section.

5. Comparison of Theory and Experiment

For comparison of the analytical results with the available experimental data, let a new parameter Z^* be introduced; where $a\cdot Z^*$ is defined to be one-half the maximum rise of the fluid at $\phi = 0$ minus the maximum drop at $\phi = 0$. Hence,

$$Z^*(r) = \frac{1}{2a}\left[\bar{Z}\left(\omega t = \frac{\pi}{2}, \phi = 0, r\right) - \bar{Z}\left(\omega t = \frac{3\pi}{2}, \phi = 0, r\right)\right] \tag{38}$$

Dodge et al. presented experimental results obtained for a sinusoidally excited tank having a complete circular cross section. Results obtained for a sinusoidally excited quarter tank were given by Kana.[16]

It should be noted that for a complete circular cross section, α is equal to one-half and the origin of the cylinder coordinate ϕ is determined by imperfections in the tank construction, excitation, etc. Setting α equal to one corresponds to a circular cross section with a rigid plate stretching from the inner wall to the outer wall along the plane $\phi = 0$. For a quarter tank, α is equal to one-fourth.

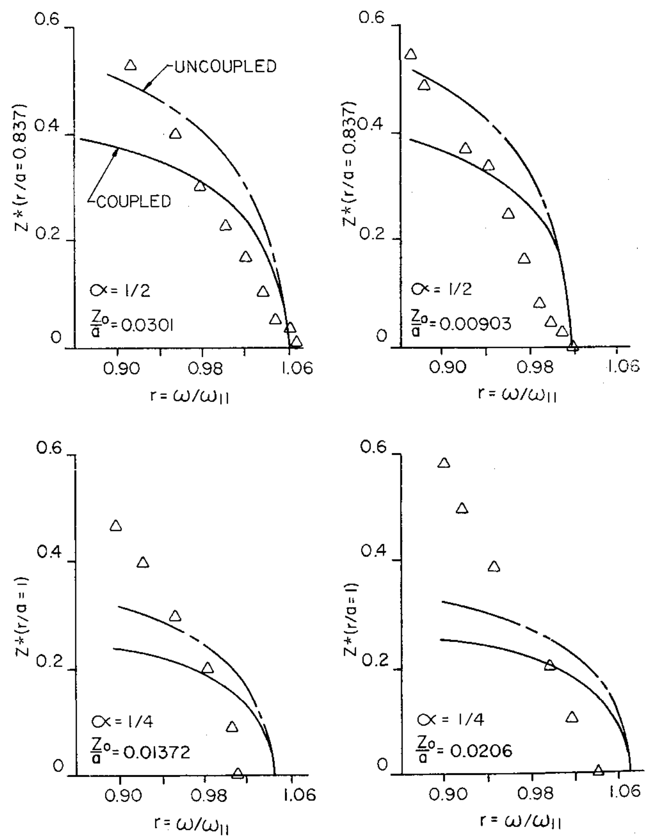

Fig. 7 Theoretical and experimental results for the one-half subharmonic response in the (1,1) mode. (b/a = 0, h/a = 2, Δ denotes experimental points).

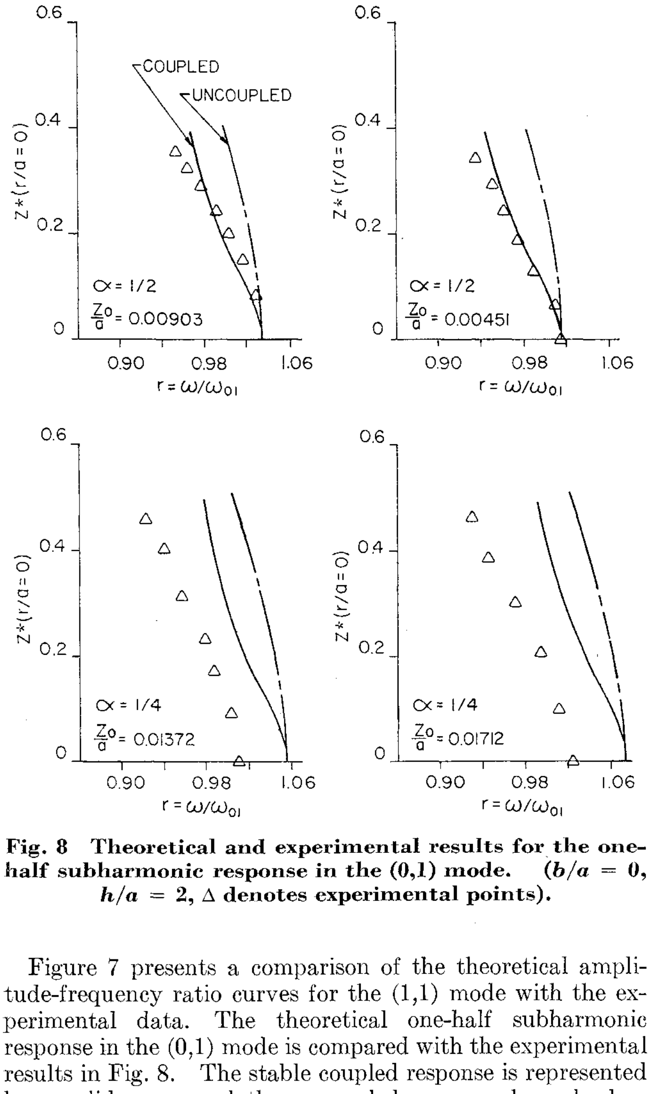

Fig. 8 Theoretical and experimental results for the one-half subharmonic response in the (0,1) mode. (b/a = 0, h/a = 2, Δ denotes experimental points).

Figure 7 presents a comparison of the theoretical amplitude-frequency ratio curves for the (1,1) mode with the experimental data. The theoretical one-half subharmonic response in the (0,1) mode is compared with the experimental results in Fig. 8. The stable coupled response is represented by a solid curve and the uncoupled response by a broken curve. In locating the experimental points, the experimental natural frequencies were used.

For the (0,1) mode response, the coupled solution gives better comparison than the uncoupled throughout the range of fluid amplitudes. For the (1,1) mode response, the coupled solution is somewhat better at lower amplitudes and the uncoupled solution prevails at higher fluid amplitudes.

The effect of damping is to shift the response curves to the left. This would particularly improve the comparison at the lower amplitudes. The inclusion of higher order terms in the equations of motion would be expected to improve the theoretical results at higher fluid amplitudes.

6. Discussion of the Results and Conclusions

A theoretical analysis of the fluid behavior in a longitudinally excited tank of arbitrary sector-annular cross section has been presented. Depending on the excitation parameters and initial disturbances, the free surface will maintain a planar shape or develop waves which oscillate as a subharmonic, harmonic, or superharmonic of the excitation frequency.

A nonlinear formulation was required to describe the finite amplitude motion. It was shown that because of stability considerations, a finite amplitude response having a frequency $N/2$ times the excitation frequency can be maintained only if the disturbances are of order greater than $0[(Z_0/a)^N]$.

An uncoupled analysis is sufficient to predict the general nature of the fluid behavior. Because of the computational difficulties, it is suggested that such an analysis be the initial phase in the investigation of any nonlinear fluid oscillation problem.

Even though the fluid container considered was of a particular class, it is believed that the determined characteristics of the fluid motion are common to other longitudinally excited tanks. Detailed investigation of motion in cylinder tanks with different cross sections may be made by using the corresponding eigenfunctions instead of the cylinder functions. Furthermore, the method utilized for simplifying the nonlinear free surface conditions is applicable to other types of tank excitation.

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